The Search for Pattern: Student Understanding of the Table of Values Representation for Function

Julie Ryan and Julian Williams

This study explores student understanding of the table of values representation for the function concept. One hundred and seventy eight students in Years 8, 9 and 10 across three schools in Melbourne undertook two pen-and-paper tests which sought to uncover the nature of the links students made between different representational forms for function. For the table of values representation there are many students who are operating in an arithmetic or pre-algebraic frame across the three year levels. There are low levels of response for algebraic patterning and indications that most students are at a rhetorical or syncopated stage of symbolism.

The Function Concept: Understanding of Symbols and Structure

The function concept is a crucial one in secondary school mathematics as it can be seen as a unifying idea throughout the algebra curriculum. The function concept ultimately requires the student to understand *structure*. This conceptual understanding requires a significant step from numerical processing and it is the bridge between arithmetic and algebra that is a focus of different introductory perspectives to school algebra. The four representations for function promoted in current curricula involve natural language description, symbolic, graph, and table of values representations. These representations and translations between them were the focus of a larger study, but only student understanding of the table of values representation is reported here.

Through the use of *symbols* in algebra a dexterity of thought becomes possible, where mental objects can be manipulated, or can be held suspended and unresolved, at will. The compactness of symbol for idea in algebra is similar to word symbols in language but the special syntax and grammar of algebra are not always intuitive and thus present a cognitive leap for the student (Herscovics, 1989; Kaput, 1989). Symbolisation requires formal levels of thought and the encapsulation or *reification of processes as objects* requires a major step forward from arithmetic thought (Harel & Dubinsky, 1992).

In moving from arithmetic to algebra the student needs to transform implicit knowledge of rules and structure into explicit knowledge. Making an idea explicit (or explicitation in van Hiele's terms) requires abstraction, a use of symbol to represent classes of objects, and a new syntax of expression. This may be difficult for students.

From Arithmetic to Algebraic Thinking

An idea and its symbol are subject to separation. Such detachment is a powerful achievement of symbolic algebra, yet it is a trap for the learner. That the ubiquitous x has *flexible meaning* causes confusion too for the beginner. Is it a number, any number, a set of numbers, or all numbers? Can we work with it when we do not know its value? Once x becomes a conceptual entity in itself, detached from known number, the learner has algebraic power of compression of representation, and can manipulate the mental and written objects according to accepted algebraic rules, and can then return and reattach meaning within the context of a problem.

The flexibility of thought required here is considerably greater than in arithmetic, which has been the grounding for the beginning algebra student. The arithmetic frame is usually strongly established and the imperatives of student processing behaviour in arithmetic are different in nature to that required in algebra. To think algebraically is distinctly different to thinking arithmetically, yet in the transition to algebraic thinking use of the arithmetic frame and generalisation from arithmetic behaviour is sensible.

Generalisation is seen to be the defining characteristic in the shift from an arithmetic perspective to an algebraic perspective: *generalising numerical patterns* is at the heart of early algebra (Sfard, 1995). Consideration of numerical patterning in arithmetic and geometry are often seen as an intuitive root for the beginning algebra student (Bell, 1988;

Mason, 1996) and current curriculum statements are drawn to them as suitable *starting points* (Australian Education Council, 1991; National Council of Teachers of Mathematics, 1989). A notion of variability is also a significant stepping stone essential for algebraic understanding. These two concepts, generality and variability, embody the structural perspective that is the essence of algebra.

The function concept draws together the key concepts of variability and generality. The study of relations between variables is a prime focus of algebra. The dependency relationship can be demonstrated in different ways: expressed *compactly* by a symbolic rule or formula, *particularised* in a numerical table of values, or *visualised* in a graph.

The Search for Pattern: Generality and Variability

While the Australian curriculum document *A National Statement on Mathematics for Australian Schools* (AEC, 1991) states that "(m)athematics brings to the study of patterns an efficient and powerful notation for representing generality and variability, and for reducing complexity – algebra" (p. 187), the table of values representation for function is not promoted *explicitly* in any of the sub-strands as a medium for understanding the basic concepts of algebra, namely generality and variability.

The different representations for the function concept listed in the AEC document each have their own structure and therefore require a different perspective of the student. This study sought broadly to answer the questions: (1) What is the nature of student understanding of the function concept? (2) What is the nature of student understanding of each of the representations for function? (3) What connections are made across the representations? (4) What are the implications of the findings for teaching?

With these broad questions in mind the role of verbal description (so-called natural language) and understanding of the tabular representation for function were seen to be of particular interest in the light of limited related research. Calls for building on student numerical patterning skills for introductory algebra (AEC, 1991) suggested that an examination of that ability and its relation to algebraic development was required.

Natural Language, Table of Values and Symbolic Algebra

In order to capture the underlying function rule for a table of values a student may use a natural language description, for example 'double the number and add one' or 'add four to the number'. In attempting to move the student to *symbolic* language the classroom teacher may temporarily use syncopated algebra where symbols *abbreviate* the inputs and outputs and the mapping is made more explicit: 'the output is four more than the input' to 'y is 4 more than x' to 'y is x plus 4'. Thus the relationship of input-output is moved (subtly) from comparison to process. The *syncopated algebra*, 'y is x plus 4', is sometimes used as a transitional language in the development of symbolic algebra '$y = x + 4$'. (From this point in the discussion 'natural language' will refer to this syncopated algebra common to classroom dialogue in the introductory stages of algebra.)

The historical role of rhetorical algebra (prose) and, more particularly, syncopated algebra in the evolution of symbolic algebra (Harper, 1987) demonstrated that there were significant mental stumbling blocks in moving from the cumbersome prose form of description to a fully *symbolised* algebra where new mental objects serve to compress both context and operation (Kleiner, 1989).

The table of values is less flexible and compact form than the symbolic form, yet its structure and syntax, too, need to be understood. It does not capture generality, but lists *examples* of numerical values paired under the functional relationship. The organisation and relationships of the parts of the table and its structure are different again to both natural language and symbolic algebra. The student is required to identify a set of x, a set of y, and to decipher an ordered set of arithmetical operations between a pair of x and y which holds for all x and y pairs. The generality is implied in that the relationship should also hold for extrapolation and interpolation under a relevant domain set.

For algebraic understanding of $y = x + 4$ the student must merge both structural conception and process into a conceptual entity – an algebraic procept – and be able to think flexibly so that either concept or process can be invoked as required (Gray & Tall, 1994): the duality of the function concept must be managed as a dichotomy of relationship and process *at will.* The student requires flexibility of perspective. Similarly, for the table of values the student needs to develop a proceptual view of the table as both relationship and process.

Research Methodology and the Test Items

For this study two written tests were formulated to investigate student understanding of key sub-concepts of the function concept across the four representations. The first test focussed on the *table of values representation* for function and the second test focussed on the *graphical representation* for function. The overall context for both tests was the linear function. The year levels chosen for study were years 8, 9, and 10 in secondary schools. The tests were undertaken at the end of the school year in October, November and December, and the age range of the students was 13 to 16 years of age. A sample of 178 students across years 8, 9, and 10 in three schools was considered to be appropriate to capture a representation of the most common responses to the test items. Each of 9 class groups was visited twice in their own school environment and each class was asked to complete two written tests each of one hour duration over a period of 4 to 6 weeks.The schools in the sample came from the government school sector in the Australian State of Victoria and had quite different profiles in terms of students and academic resources. This sample was designed to provide a variety of responses from a broad spectrum of students' progress in understanding the function concept. The three participating schools provided a broad sample of Victorian students in terms of background, achievement and opportunity.The test items discussed here were designed to reveal students' understanding of functions represented in tables. Students were asked to state a relationship or rule between the numbers in the x and y columns of the given tables, firstly in natural language (Da), then in algebraic form (Db), and lastly to supply y values for given x values (Dc). These items were designed to assess: (1) Which structural features of a mathematical table of values are recognised by students? (2) What patterns are seen by students and which ones are debilitating in attaining an algebraic view? (3) Whether some students are in a rhetorical stage of symbolism, (4) Which rules present a more difficult translational task, and (5) What are the error phenomena?

D4

x	y
1	3
2	5
3	7
5	11

(a) Describe the relationship or rule between the numbers in the x and y columns in words:

(b) The rule is y =

(c) If these also belonged to the table, fill in the correct y values to match the x values.

x	y
-1	
0	
10	

Figure 1. Items Da4, Db4, Dc4

The sequence of linear functions in section D was: (D1) $y = x + 3$, (D2) $y = 4x$, (D3) $y = x$, (D4) $y = 2x + 1$, (D5) $y = 5 - x$, and (D6) $y = x - 5$. Items D1, D2, D3 and D6 are uni-operational functions in the sense that one arithmetical operation is performed on the independent variable, and numbers D4 and D5 are bi-operational in the sense that two operations are performed on the independent variable. The six linear functions for Da, Db and Dc were chosen to represent suspected different levels of difficulty. The first four questions (D1 to D4) all had x = 1, 2, 3, 5 in the initial table with their matching y values. The omission of x = 4 was meant to break the pattern of x-change in case a vertical cue was being used by the student. The last two questions (D5 and D6) had a different sequence of the independent variable values: D5 had x = 1, 2, 3, 4 to monitor the effect of the distracting pattern in the y column and D6 had x = 7, 4, 2, 1 to monitor the effect of a decreasing change in the x values.

Analysis

In the analysis of student responses a tree diagram was used to show the connections and stumbling blocks between the three tasks of each item of section D. The numbers in the tree diagram below (Figure 2) represent the numbers of students correct (c) or not correct (nc) for each task of D4. Thus the left-most path in the tree diagram labelled as path (1,1,1) represents those students who successfully responded to all tasks (verbal - Da4, algebraic - Db4 and numeric - Dc4). The next path in the sieve, labelled as (1,1,0), represents those students who were correct on both Da4 and Db4 but not correct on Dc4.

One interesting path through different patterning abilities is detailed by arrows in the figure. This path, described as path (0,0,1), represents those students who were unsuccessful in both Da4 (verbal description) and Db4 (symbolic algebraic description), but were successful in Dc4 (numeric patterning). These students perhaps had some strategy other than verbal description or algebraic description which allowed them to respond correctly on the final task. The path is characterised as *'successful inarticulate patterning'*.

The eight paths are characterised as: (1,1,1) successful articulate algebraic patterning; (1,1,0) unsuccessful articulate algebraic patterning; (1,0,1) successful rhetorical patterning; (1,0,0) unsuccessful rhetorical patterning; (0,1,1) successful articulate algebraic patterning; (0,1,0) unsuccessful articulate algebraic patterning; (0,0, 1) successful inarticulate patterning; (0,0,0) non-patterning. (Path (0,1,1) is characterised in the same way as (1,1,1): successful articulate algebraic patterning. Similarly, path (0,1,0) is characterised in the same way as (1,1,0): unsuccessful articulate patterning. The assumption is made that natural language description ability is subsumed in algebraic description ability.)

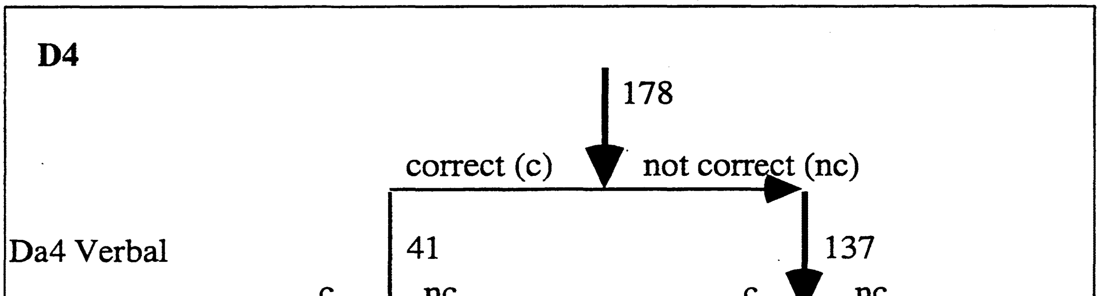

Figure 2. Numbers of students successful on each task of item D4, $y = 2x + 1$

In all six questions it appears that there are students who can complete a table of values and yet are not able to correctly express the underlying function in natural language or symbolic language. Their path is characterised above as 'successful inarticulate patterning' (0,0,1). For this group the pattern within the table is accessible in some other way. Table 1 below indicates the number of students in this category for the different table functions.

Item	Table function	Number of students (n = 178)
D1	y = x + 3	22
D2	y = 4x	10
D3	y = x	15
D4	y =2x + 1	4
D5	y = 5 – x	19
D6	y = x – 5	12

Table 1. 'Successful inarticulate patterning' for different functions

The three tasks in each item were used to form three scales and a reliability analysis was undertaken on SPSS (no. of items = 6, no. of cases = 178) for each of the scales. Statistical analysis shows that there is a strong positive correlation for each item with Scale Da (except for item Da5 where there is a weak positive correlation). That is the items contribute reliably to the scale measure Da of 'verbal patterning ability' (Cronbach-alpha = 0.777). There is a strong positive correlation for each item with Scale Db (except for item Db5 where there is a weak positive correlation). That is the items contribute reliably to the scale measure Db of 'algebraic patterning ability' (Cronbach-alpha = 0.851). There is a strong positive correlation for each item with Scale Dc (except for item Dc5 where there is a weak positive correlation). That is the items contribute reliably to the scale measure Dc of 'numeric patterning ability' (Cronbach-alpha = 0.786).

The D5 items had the weakest positive correlations with each of the three scales. The table function for these items was $y = 5 - x$. This function can be seen to be different from the others in its presentation: the given table is shown in Figure 3 below. The entries as given proved to be overwhelming distracters from the (horizontal) function pattern where students were drawn rather to the vertical patterns in the x and y columns.

The down/up *parallel patterning* distracter in item Dc5 resulted in 12.4% of students responding as shown in Figure 3 below for $y = 5 - x$. This response is a non-algebraic mapping where students have disregarded the concept of correspondence between variables.

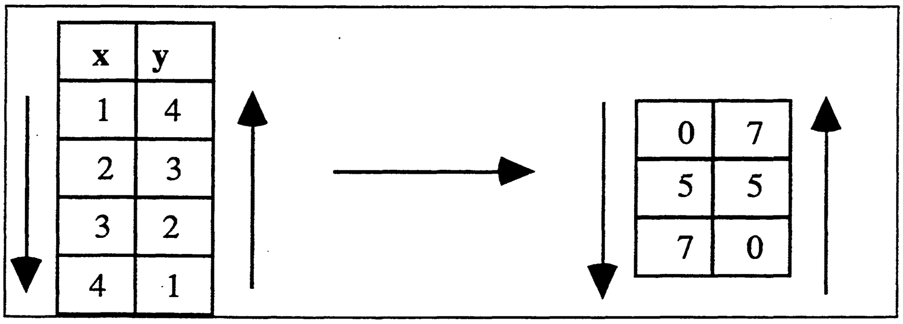

Figure 3. Item Dc5: Parallel table patterning

Facility and Error Types

Verbal patterning: As shown in Table 2 below the functions in order of ease of *verbal description* were $y = x$, $y = x + 3$, $y = 4x$, $y = x - 5$, $y = 2x + 1$, $y = 5 - x$. For translation from table to verbal description, the identity function is the easiest, and the additive function is easier than the multiplicative function. The uni-operational functions are easier than the bi-operational functions.

Item	Function	Facility (%)	Error types
Da1	y = x + 3	60.7	reversal
Da2	y = 4x	46.6	reversal
Da3	y = x	62.9	
Da4	y =2x + 1	23.0	horizontal additive patterning
Da5	y = 5 – x	2.8	down/up parallel patterning
Da6	y = x – 5	34.3	reversal

Table 2. Facilities and error types for 'verbal patterning ability'

The identity function (Da3) was variously described correctly as (y is) 'the same' (as x) or 'one times', the additive function was described correctly as 'add 3', the multiplicative function mostly described as y is '4 times' x but a small number of students used a (correct) notion of double add double, that is $y = 2x + 2x$. The subtractive function was mostly described as 'take away 5'. The much harder bi-operational function, $y = 2x + 1$, (facility 23%) was described mostly as 'double and add 1', but it was susceptible to incorrect *horizontal additive patterning*. The 12.4% of students who were drawn to this patterning described the relationship or rule between the numbers in the x and y columns in terms of a sequence of additions between each of the four pairs of (x, y) given, that is 'add 2, add 3, add 4, add 6'. They were unable to translate this to either algebraic rule or apply it correctly to the table items in Dc4. Thus a dependence on an additive perspective of table patterning constitutes a cognitive obstacle.

The most difficult function to describe verbally was $y = 5 - x$ (facility 2.8%). The function is bi-operational in terms of 'add 5 to the negative of x', however it was expected that most students would describe the relationship as 'x plus y is 5'. This was not the case. As discussed above there was a powerful distracter at work. 27.0% of students (across all year levels) who were drawn to the vertical *down-up parallel pattern* of the x and y columns.

Algebraic patterning: There was also a very small number of students (n = 6) who *reversed* the relationships. This was pertinent to items Da1, Da2, and Da6. The other type of error for verbal items was reference to the vertical patterning of the table in terms of the incremental additive change in x and y. Apart from the 48 students (27%) drawn to this patterning for $y = 5 - x$ discussed above, there were 11 students who used the same strategy on other tables as well. Of these 11 students, 5 of them were only correct in the identification of the identity function and the additive function, but incorrect for all other tables, while the other 6 were incorrect on all other items. Their usage of this *vertical additive patterning* then can be seen to be persistent and debilitating.

The items of Db(1-6) constituted a reliable scale measure of 'algebraic patterning ability' as indicated above. As shown in Table 3 below the functions in order of ease of algebraic description is the same order as for 'verbal patterning ability'. The most difficult function to describe algebraically was $y = 5 - x$. This item had a dramatically poor facility of 2.3% with most students omitting it. As discussed in the previous section relating to verbal patterning the problem lay in the distraction of the parallel reverse patterning in the x and y columns.

Item	Function	Facility (%)	Error types
Db1	y = x + 3	39.3	verbal, process
Db2	y = 4x	36.0	verbal, process
Db3	y = x	47.2	verbal, process
Db4	y =2x + 1	20.2	verbal, single pair
Db5	y = 5 - x	2.3	verbal, process
Db6	y = x - 5	27.5	verbal, process

Table 3. Facilities and error types for algebraic patterning ability

The largest numbers of incorrect responses were from students who gave a verbal rule without algebraic symbolism. The next largest group attempted algebraic symbolism but failed to include the independent variable in the statement. This group concentrated on the *process* or action. For $y = x + 3$, 12.9% of students wrote y = +3, or y = 3 times larger; for $y = 4x$, 7.3% of students wrote y = multiply by 4 or y = 4; for $y = x$, 8.4% of students wrote y = 0, or y = +0, or y = 1, or y =×1; for $y = 5 - x$, 2.8% of students wrote y = 5; and for $y = x - 5$, 5.6% wrote y = 5 or -5.

A small number of students (5.1%) drew algebraic patterning from *a single pair* of data for the bi-operational function $y = 2x + 1$. Here the algebraic rule was given as y = x + 2, or y = +2, or y = 2. Students here noticed that the elements of the first pair in the table (1, 3) had a difference of 2. The numbers of reversals in algebraic patterning were minimal.

Numeric patterning: The items of Dc(1-6) constituted a reliable scale measure of 'numeric patterning ability' as indicated. As shown in Table 4 below the functions in order of ease of verbal description is almost the same order as for verbal and patterning abilities with Dc1 and Dc3 interchanged. However, the facilities for numeric patterning are higher than for the previous abilities except for question D4.

Item	Function	Facility (%)	Error types
Dc1	y = x + 3	74.2	
Dc2	y = 4x	49.4	arithmetic structure mapping, zero, -1
Dc3	y = x	70.8	
Dc4	y =2x + 1	17.4	arithmetic structure mapping, zero, -1
Dc5	y = 5 - x	12.9	arithmetic structure mapping, down/up parallel patterning
Dc6	y = x - 5	41.6	arithmetic structure mapping

Table 4. Facilities and error types for 'numeric patterning ability'

The responses here uncovered further detail of students' patterning perspectives as some students mapped an 'arithmetic structure' from the initial table to the new entries rather than the function structure. For example, for y = 2x + 1 some students (11.2%) extracted the table pattern as +2, +3, +5 from the first three entries, and mapped this on to the new entries giving (-1, 1), (0, 3), and (10, 14), or extracted the pattern as +2 from the first pair to give (-1, 1), (0, 2), and (10, 12). These students have failed to recognised the underlying algebraic structure of the table and have taken no account of the *order* of the entries or correspondence of the *variables*.

Conclusions

The research questions considered the connections students make between the table, natural language and symbolic algebra. The analysis here indicated that many students did not recognise the structural features of the mathematical table of values for the linear functions used here. What were termed as 'an arithmetic structure mapping', 'a vertical additive patterning' and 'a horizontal additive patterning' indicated a non-recognition that two sets of values of corresponding variables were *co-related* in the same way – that there was a generality of *correspondence*. Students who responded with these perspectives can be seen to be responding pre-algebraically.

The parallel patterning behaviour in D5 dramatically showed that other patterns in tables are a distraction if an algebraic view is not strongly established. Similarly a vertical additive patterning which paid attention to incremental change on x and y without recognition of correspondence was demonstrated as persistent and debilitating.

A trace of the various paths for successful and unsuccessful translations between the three tasks showed that there were substantial numbers of students who were not able to supply algebraic symbolism but who were strong on verbal and numeric patterning ability. These students came from all the year levels. Their responses were termed 'successful rhetorical patterning'. Their responses on the algebraic tasks were sometimes partially symbolised thus constituting a syncopated stage of symbolism.

There were students who were able to supply the numeric values but who were unable to express the pattern of the table verbally or algebraically. Their responses were termed 'successful inarticulate patterning' and students exhibiting this response were found to be persistent in this response type. There were also large numbers of responses termed 'non-patterning' indicating that for many students the function concept is not established at all. Reversal errors occurred in all translations but were most common in the verbal description of the table pattern. There were problems with processing zero and negative one in the numeric supply tasks. Arithmetic perspectives for table patterning were indicated in the three translations in drawing the table pattern from one pair of data, from vertical patterning alone, or from a sequence of operations. These errors can be seen to arise from an action or process conception of function (Harel & Dubinsky, 1992).

Bibliography

Australian Education Council (AEC) (1991). *A national statement on mathematics for Australian schools.* Carlton: Australian Education Council and Curriculum Corporation.

Bell, A. (1988). Algebra: Choices in curriculum design. In A. Borbas (Ed.), *Proceedings of the 12th International Conference for the Psychology of Mathematics Education, Vol 1* (pp. 147-153). Hungary: OOK.

Dienes, Z. P. (1960). *Building up mathematics.* London: Hutchinson Educational. (Revised edition, 1969).

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A 'proceptual' view of simple arithmetic. *Journal for Research in Mathematics Education, 25* (2), 116–140.

Harel, G. & Dubinsky, E. (Eds) (1992), *The concept of function: Aspects of epistemology and pedagogy.* (MAA Notes no. 25). Washington DC: Mathematical Association of America

Harper, E. (1987). Ghosts of Diophantus. *Educational Studies in Mathematics 18,* 75-90.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner and C. Kieran (Eds), *Research issues in the learning and teaching of algebra* (pp. 60-86). Lawrence Erlbaum Associates: NCTM.

Kaput, J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner & C. Kieran (Eds), *Research issues in the learning and teaching of algebra,* (pp. 167-194). Reston, VA: NCTM.

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *College Mathematics Journal 20* (4), 282-300.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee. (Eds), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Dordrecht: Kluwer Academic Publishers.

Rojano, T. (1996). Developing algebraic aspects of problem solving within a spreadsheet environment. In N. Bednarz, C. Kieran, & L. Lee. (Eds), *Approaches to algebra: Perspectives for research and teaching* (pp. 137-145). Dordrecht: Kluwer Academic Publishers.

Schaaf, W. L. (1930). Mathematics and world history. *Mathematics Teacher, 23,* 496-503.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior, 14,* 15-39.

Note: The authors email addresses: J.T.Ryan@mmu.ac.uk (Manchester Metropolitan University) and JWilliams@fs1.ed.man.ac.uk (University of Manchester)